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TORSIONAL-MODE TRANSIENTS IN VARIABLE-THICKNESS SHELLS OF REVOLUTION

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Abstract—For thin shells of revolution the existence of torsional-vibration modes, uncoupled from bending and extensional modes, has been established[1]. Here a linear second-order differential equation for the uncoupled torsional stress mode is obtained and its solution for impact loading of shells is sought. The mode-superposition method which utilizes the natural modes of vibration predicted by elementary theory, is, in general, not satisfactory for sharp impact loading as many modes are often required for convergence. Hence we employ two novel techniques for solving the impact problems. Firstly a formal asymptotic procedure, based on extensions to geometrical optics, is employed to generate asymptotic wavefront expansions. Rigorous justifications for this formal technique are provided in an appendix. Secondly a transform technique whereby solutions are sought in terms of Bessel functions is discussed and applied to particular impact loading problems. The Bessel function solutions found here can be used to determine the natural frequencies of the shells. Shells both finite and infinite in extent are discussed and reflections at a stress-free end are examined.

1. INTRODUCTION

Very little research into the dynamic loading of variable-thickness shells of revolution has appeared in the literature. The motivation for examining such problems is to be found in the area of space research where these structures are employed in the design of missiles and allied systems. Aspects of vibrations in shells of uniform thickness have been studied by Mathieu[2], Love[3], Lamb[4], and more recently by Garnet *et al.*[1] while variable-thickness shells have been discussed by Soni *et al.*[5], and more recently by Ramamurti and Ganesan[6].

The present study is concerned with sharp impact problems for variable-thickness shells of revolution. For reasons alluded to in the abstract we must find alternatives to the mode-superposition method often employed in shell analysis.

Our first alternative is a formal asymptotic procedure that has been developed principally by J. B. Keller *et al.* at the Courant Institute in New York[7]. By this procedure one assumes a particular asymptotic form for the solution of an impact problem and then determines ordinary differential equations which when solved give the various components in the asymptotic expansion. For a detailed discussion of this asymptotic method (and a list of related references), as it applies to waves in solid media, see[8]. Because this method is formal and no general proofs for its validity have been published, we will employ rigorous arguments in Appendix A and give partial justification for its validity in the present context.

The second method used to analyze our impact problems was originally devised by Eason[9] for finding solutions in terms of Bessel functions to problems of wave propagation in elastic media. By assuming a solution in terms of Bessel functions Eason reversed the usual procedure and was then led to the most general forms that the elastic coefficients could assume to give such a solution.

2. FORMULATION

Throughout this paper we will adhere to the notation in Novozhilov[10]. For the orientation of forces and moments see Figs. 8 and 9 on page 32 of this reference.

We consider an element cut from a shell of revolution. The curvilinear coordinates α_1 and α_2

are lines of principal curvature of the middle surface in the undeformed shell. The z-axis is taken normal to this surface. Corresponding to these orthogonal axes α_1, α_2 and z are the displacements u, v, and w, respectively.

Assuming that the displacements u, v and w are functions of the single meridional coordinate α_1 yields the specialized stress-strain relations given by eqns (2a)–(2f) of [1]. As we are interested in the torsional-mode waves we reproduce the relevant stress-strain eqn (2c) of [1], i.e.

$$T_{12} = T_{21} = \frac{E\delta}{2(1+\mu)}\left[\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v}{A_2}\right)\right], \tag{2.1}$$

where E is Young's modulus, $\delta(\alpha_1)$ is the variable shell thickness, μ is Poisson's ratio, and $A_1(\alpha_1)$ and $A_2(\alpha_1)$ are metric coefficients given by

$$(ds)^2 = A_1^2(d\alpha_1)^2 + A_2^2(d\alpha_2)^2. \tag{2.2}$$

Following Garnet *et al.* [1] and replacing the body forces by the inertia terms in the equations of equilibrium [10] gives eqns (4a)–(4e) of [1]. The relevant equation for our purposes is (4b) which we reproduce here, i.e.

$$\frac{1}{A_1A_2^2}\frac{\partial(A_2^2T_{12})}{\partial\alpha_1} - \rho\delta\frac{\partial^2 v}{\partial t^2} = 0, \tag{2.3}$$

where ρ is the mass density per unit thickness. Now, differentiating in (2.1) and combining the result with (2.3) we find the stress equation of motion to be

$$\frac{\partial}{\partial\alpha}\left[\frac{1}{\delta A_1A_2^3}\frac{\partial}{\partial\alpha}(A_2^2T)\right] - \frac{2A_1\rho(1+\mu)}{A_2E\delta}\frac{\partial^2T}{\partial t^2} = 0, \tag{2.4}$$

where we have introduced the notation

$$T_{12} \equiv T(\alpha, t),\ \alpha_1 \equiv \alpha. \tag{2.5}$$

According to the formal asymptotic procedure, we seek solutions of (2.4) in the form

$$T \sim \sum_{\nu=0}^{\infty} T_\nu(\alpha)F_\nu[t - L(\alpha)],\ T_{-1} = T_{-2} = \cdots = 0, \tag{2.6}$$

where the F_ν's are related by

$$F'_\nu = F_{\nu-1},\ \nu = 1, 2, \ldots. \tag{2.7}$$

The prime in (2.7) refers to differentiation with respect to the entire argument $(t - L)$, and (2.7) enables us to relate all of the F_ν's to F_0 (the *waveform*) by successive integrations. For example, if F_0 is the Heaviside unit function, $U(t)$, defined by

$$U(t) = 1 \text{ for } t > 0,\ U(t) = 0 \text{ for } t < 0,$$

then

$$F_\nu = (t - L)^\nu U(t - L)/\nu!. \tag{2.8}$$

Note that F_0 vanishes for negative argument, i.e. in front of the *wavefront* whose equation is given by $t = L(\alpha)$, where L is called the *phase* or *eikonal* function. For F_ν given by (2.8), the coefficients T_ν in (2.6) are the jump conditions for the stress and its derivatives across the wavefront.

Our second technique is applied after (2.4) has been transformed into a wave equation with variable speed. The Laplace transform

$$\phi(x,p)=\int_0^\infty e^{-pt}\sigma(x,t)\,dt \tag{2.9}$$

where σ is related to the stress, is then employed to remove the time variable and the solution of the resulting ordinary differential equation is assumed to have the form

$$\phi(x,p)=A(p)f(x)F(w),\ w=pg(x). \tag{2.10}$$

As a result of this assumption conditions on the elastic parameters, thickness, and metric coefficients are discovered which lead to solutions in terms of Bessel functions.

3. FORMAL ASYMPTOTIC SOLUTIONS

(a) *General solution*

We consider torsional-mode stress waves whose propagation is governed by (2.4) and seek the solution in the form (2.6). Substituting (2.6) in (2.4) and carrying out the analysis outlined in [8], we obtain

$$(L')^2=2\rho A_1^2(1+\mu)/E=1/c^2, \tag{3.1}$$

and

$$2L'T_\nu'+L''T_\nu+4(A_2'/A_2)L'T_\nu-[(\delta A_1A_2^3)'/(\delta A_1A_2^3)]L'T_\nu=\mathscr{L}T_{\nu-1},\ \nu=0,1,\ldots, \tag{3.2}$$

where primes refer to ordinary differentiation with respect to α, c is the propagation speed, and

$$\mathscr{L}f(\alpha)\equiv f''+4(A_2'/A_2)f'-[(\delta A_1A_2^3)'/(\delta A_1A_2^3)]f'+2(A_2'/A_2)^2f$$
$$+2(A_2''/A_2)f-2[(\delta A_1A_2^3)'A_2'/(\delta A_1A_2^3)A_2]f. \tag{3.3}$$

Equation (3.1) is the well-known eikonal equation of geometrical optics. Integrating this equation along a ray associated with the wave motion, we have

$$L(\alpha)=\bar{L}\pm\int_{\bar{\alpha}}^{\alpha}d\tau/c(\tau),\ \bar{L}\equiv L(\bar{\alpha}), \tag{3.4}$$

where the upper and lower signs are associated with waves travelling in the direction of increasing and decreasing α, respectively. Equation (3.4) enables us to determine the phase at any point on a ray in terms of its value at $\alpha=\bar{\alpha}$. Equation (3.2) is the transport equation and its general solution is

$$T_\nu(\alpha)=\bar{T}_\nu\left[\frac{\bar{A}_2c(\alpha)\delta(\alpha)A_1(\alpha)}{A_2(\alpha)\bar{c}\bar{\delta}\bar{A}_1}\right]^{1/2}\pm\frac{1}{2}\int_{\bar{\alpha}}^{\alpha}\left[\frac{c(\alpha)\delta(\alpha)A_1(\alpha)A_2(\tau)c(\tau)}{A_2(\alpha)\delta(\tau)A_1(\tau)}\right]^{1/2}\mathscr{L}T_{\nu-1}(\tau)\,d\tau, \tag{3.5}$$

where the bar indicates that the corresponding function is evaluated at $\alpha=\bar{\alpha}$ and the convention of signs is the same as for (3.4).

We have now, in principle, obtained an asymptotic representation for the solution of (2.4). This asymptotic representation is given by (2.6) with the phase $L(\alpha)$ and amplitude functions $T_\nu(\alpha)$ determined from (3.4) and (3.5). The boundary conditions $\bar{L}$ and $\bar{T}_\nu$ may be determined in a straightforward fashion from the data of the problem. If the waveform is given by $F_0=U(t)$, then the asymptotic representation has the form

$$T\sim\sum_{\nu=0}^{\infty}T_\nu(\alpha)\frac{[t-L(\alpha)]^\nu}{\nu!}\mathrm{U}[t-L(\alpha)], \tag{3.6}$$

whereas if $F_0=\mathscr{D}(t)$, where $\mathscr{D}(t)$ is the Dirac delta function, or, to be more precise, the measure representing a unit mass concentrated at the origin, then it has the form

$$T \sim T_0(\alpha)\mathscr{D}[t - L(\alpha)] + \sum_{\nu=1}^{\infty} T_\nu(\alpha)\frac{[t - L(\alpha)]^{\nu-1}}{(\nu-1)!} U[t - L(\alpha)]. \tag{3.7}$$

(b) *Reflections*

We now outline briefly how the formal asymptotic technique may be modified to treat reflections in a finite shell of revolution. We consider a finite shell of revolution, $a \le \alpha \le b$ $(a > 0)$, open at both ends. At $\alpha = \bar{\alpha} = a$, of the initially quiescent shell, we apply an impact in the form of a torsional stress suddenly applied and thereafter steadily maintained, while the end $\alpha = b$ is maintained in a stress-free state. Thus

$$\left.\begin{aligned} \bar{T} &= A\, U(t),\ \alpha = \bar{\alpha} = a, \text{A constant,} \\ \bar{T} &= 0, \qquad \alpha = \bar{\alpha} = b. \end{aligned}\right\} \tag{3.8}$$

A transient stress wave is produced which leaves the end $\alpha = a$ at $t = 0$ and travels with velocity $c(\alpha)$ in the direction of increasing α, i.e. an *outgoing wave*. We will denote this outgoing wave by T^1 and its phase and amplitude functions by L^1 and T_ν^1. It has a representation of the form (3.6) where L^1 and T_ν^1 are determined from (3.4) and (3.5) taken with the upper signs and in conjunction with the appropriate boundary conditions[8]

$$\bar{T}_\nu^1 = \begin{cases} A & \text{if } \nu = 0, \\ 0 & \text{if } \nu > 0 \text{ or } \nu < 0, \end{cases} \tag{3.9}$$

$$\bar{L}^1 = 0,\ F_0 = U(t),\ \bar{\alpha} = a.$$

Up until time $t = L^1(b)$ the solution is given by the outgoing wave $T^1(\alpha, t)$.

We now assume that at the stress-free boundary $\alpha = b$ there is produced a reflected wave, T^2, which has the representation

$$T^2 \sim \sum_{\nu=0}^{\infty} T_\nu^2(\alpha) F_\nu[t - L^2(\alpha)],\ T_\nu^2 \equiv 0 \text{ for } \nu < 0. \tag{3.10}$$

The amplitude functions, T_ν^2, satisfy the transport eqn (3.2) and the phase function, L^2, satisfies the eikonal eqn (3.1). The solution of (3.2) for a wave travelling from $\alpha = b$ in the direction of decreasing α is given by (3.5) taken with the lower sign and $\bar{\alpha} = b$. We justify our assumption that a reflected wave is produced at $\alpha = b$ by demonstrating that the boundary condition of zero stress can be satisfied by[7]

$$T = T^1 + T^2. \tag{3.11}$$

This is indeed the case and gives rise to the boundary conditions

$$\bar{T}_\nu^2 = -\bar{T}_\nu^1,\ \bar{L}^2 = \bar{L}^1,\ \alpha = \bar{\alpha} = b. \tag{3.12}$$

Then, because $\bar{T}_\nu^1$ and $\bar{L}^1$ are known we may completely determine the reflected wave $T^2(\alpha, t)$.

(c) *Sample problem A*

We consider an initially quiescent conical shell $(a \le \alpha < \infty,\ a > 0)$ of semivertex angle ϕ whose variable thickness is given by the relationship

$$\delta = \delta_0(\alpha/a)^N;\ \delta_0, N \text{ are constant,} \tag{3.13}$$

to have its end $\alpha = a$ subjected to an impact in the form of a torsional stress suddenly applied and thereafter steadily maintained. Here

$$A_1 = 1,\ A_2 = \alpha \sin\phi, \tag{3.14}$$

and the boundary conditions $\bar{T}_\nu^1$, $\bar{L}^1$ together with the waveform are given by (3.9).

Employing the boundary conditions in (3.4) we obtain the phase function

$$L^1(\alpha) = [2\rho(1+\mu)/E]^{1/2}(\alpha - a). \tag{3.15}$$

Then, from (3.5), (3.13) and (3.14) we may prove by induction that the general form for the amplitude functions T_ν^1 is given by

$$T_\nu^1 = \left(\frac{\alpha}{a}\right)^{(N-1)/2} \sum_{j=0}^{\nu} a_{j\nu}\left(\frac{\alpha}{a}\right)^{-j}, \tag{3.16}$$

where the $a_{j\nu}$ are constants given recursively by

$$a_{j\nu} = \begin{cases} -\dfrac{c}{2a}\left[\left(\dfrac{2j-1-N}{2}\right)\left(\dfrac{2j-1+N}{2}\right) - 2(2+N)\right]a_{j-1,\nu-1}/j \text{ if } 1 \le j \le \nu, \\ \dfrac{c}{2a}\displaystyle\sum_{j=1}^{\nu}\left[\left(\dfrac{2j-1-N}{2}\right)\left(\dfrac{2j-1+N}{2}\right) - 2(2+N)\right]a_{j-1,\nu-1}/j \text{ if } j=0, \nu>0, \\ A \text{ if } j=\nu=0, \\ 0 \text{ if } j<0 \text{ or } j>\nu, \end{cases} \tag{3.17}$$

where $a_{j,\nu} \equiv a_{j\nu}$. Thus the asymptotic wavefront expansion for the outgoing wave is

$$T^1 \sim \left(\frac{\alpha}{a}\right)^{(N-1)/2} \sum_{\nu=0}^{\infty} \frac{[t - L^1(\alpha)]^\nu}{\nu!} \sum_{j=0}^{\nu} a_{j\nu}\left(\frac{\alpha}{a}\right)^{-j} U[t - L^1(\alpha)], \tag{3.18}$$

where the $a_{j\nu}$ are given by (3.17) and L^1 by (3.15). The expansion in (3.18) is not valid as $\alpha \to \infty$ for $N > 0$ for them the assumption of a thin shell is violated. It is, however, applicable to the range $\alpha \le X < \infty$ for $N > 0$. If we put $N = 0$ in the above we then have the wavefront asymptotic expansion for an outgoing wave in a cone of constant thickness. This expansion is then valid for all α.

We now consider the above impact for a finite conical shell, $a \le \alpha \le b$, with the end $\alpha = b$ maintained in a stress-free state and whose thickness is again given by (3.13). Leaving the end $\alpha = a$ at $t = 0$ is the outgoing wave T^1 which has been completely determined above. The reflected wave T^2 has the expansion (3.10) whose first few terms give

$$T^2 = \{T_0^2(\alpha) + T_1^2(\alpha)(t - L^2(\alpha)) + 0[(t - L^2(\alpha))^2]\}\, U[t - L^2(\alpha)], \tag{3.19}$$

where

$$T_0^2(\alpha) = -A(\alpha/a)^{(n-1)/2}, \tag{3.20}$$

$$T_1^2(\alpha) = \left(\frac{\alpha}{a}\right)^{(N-1)/2}\left(\frac{Ac}{2a}\right)\left[\left(\frac{N-1}{2}\right)\left(\frac{N+1}{2}\right) + 2(N+2)\right]\left\{\left[1 - 2\left(\frac{a}{b}\right)\right] + \left(\frac{\alpha}{a}\right)^{-1}\right\} \tag{3.21}$$

$$L^2(\alpha) = \left[\frac{2\rho(1+\mu)}{E}\right]^{1/2}[(b-a) + (b-\alpha)]. \tag{3.22}$$

The solution T^2 is valid up until time $t = L^2(a)$ when the reflected wave T^2 has reached $\alpha = a$. A reflection occurs at $\alpha = a$ generating a reflected wave T^3 which may be constructed according to the above procedure. Any number of reflections may be handled in this manner.

(d) *Sample problem B*

We consider the above impact problem for a cylindrical shell $A_1 = 1$, $A_2 = R$, where R is its constant radius and whose thickness varies according to the law given by (3.13). Employing the methods used above we find that the asymptotic wavefront expansion for the wave which leaves $\alpha = a$ and travels in the direction of increasing α is given by

$$T^1 \sim \left(\frac{\alpha}{a}\right)^{N/2} \sum_{\nu=0}^{\infty} \frac{[t - L^1(\alpha)]^\nu}{\nu!} \sum_{j=0}^{\nu} a_{j\nu}\left(\frac{\alpha}{a}\right)^{-j} U[t - L^1(\alpha)], \tag{3.23}$$

where the $a_{j\nu}$ are given recursively by

$$a_{j\nu} = \begin{cases} \dfrac{-c}{2a}\left(\dfrac{N-2j+2}{2}\right)\left(\dfrac{N-2j-2}{2}\right)a_{j-1,\nu-1}/j \ \text{ if } 1 \le j \le \nu, \\ \dfrac{c}{2a}\displaystyle\sum_{j=1}^{\nu}\left(\dfrac{N-2j+2}{2}\right)\left(\dfrac{N-2j-2}{2}\right)a_{j-1,\nu-1}/j \ \text{ if } j = 0, \nu > 0, \\ A \ \text{ if } j = \nu = 0, \\ 0 \ \text{ if } j < 0 \text{ or } j > \nu, \end{cases} \tag{3.24}$$

and the phase $L^1(\alpha)$ is given by (3.15). This expansion is valid up until $t = L^1(b)$ when the boundary condition on $\alpha = b$ must be applied and the reflected wave determined. Because these calculations are straightforward and have been carried out for the case of a conical shell we will not include them here. Instead, we will proceed to our second method for analyzing impact problems.

4. SOLUTION BY THE LAPLACE TRANSFORM

It will be convenient to make the change of variable

$$x = \int_a^\alpha \delta(\tau)A_1(\tau)A_2^3(\tau)d\tau, \ \sigma(x, t) = A_2^2(\alpha)T(\alpha, t). \tag{4.1}$$

This transformation results in (2.4) taking the form

$$\frac{\partial^2\sigma}{\partial x^2} - \frac{2\rho(1+\mu)}{\delta^2 A_2^6 E}\frac{\partial^2\sigma}{\partial t^2} = 0, \tag{4.2}$$

which is the wave equation with variable speed. Now remove the time dependence from (4.2) with the aid of (2.9) obtaining the ordinary differential equation

$$\frac{d^2\phi}{dx^2} - \frac{2\rho(1+\mu)}{\delta^2 A_2^6 E}p^2\phi = 0, \tag{4.3}$$

where it has been assumed that σ and $\partial\sigma/\partial t$ vanish at $t = 0$.

Eason [9] has treated an equation of the same form as (4.3) and found solutions to it in terms of Bessel functions. Eason's method involves assuming a solution of (4.3) in the form (2.10). On inserting (2.10) into (4.3) it is found that F must satisfy an equation whose solution can be expressed in terms of modified Bessel functions. In addition, the parameters of the elastic media together with the thickness and metric coefficients must satisfy a certain equation which guarantees that the solution has the assumed form (2.10). We summarize the results obtained when this method is applied to the differential equation considered here.

Introduce the new variable z defined by

$$z = 1/(x + K), \tag{4.4}$$

where K is a constant and suppose that

$$\frac{2\rho(1+\mu)}{\delta^2 A_2^6 E} = z^4(C + Dz)^{2B}, \tag{4.5}$$

where B, C, and D are constants. Then, with z as the independent variable, eqn (4.3) becomes

$$\frac{d^2\phi}{dz^2}+\frac{2}{z}\frac{d\phi}{dz}-p^2(C+Dz)^{2B}\phi=0. \tag{4.6}$$

If $D\neq 0$ and $B\neq -1$, (4.6) has the solution

$$\phi=\frac{r^\eta}{z}\{C_1K_\eta(rp)+C_2I_\eta(rp)\}, \tag{4.7}$$

where

$$\eta=\frac{1}{2(1+B)},\ r=\frac{(C+Dz)^{1+B}}{D(1+B)}, \tag{4.8}$$

and $I_\eta(x)$, $K_\eta(x)$ are modified Bessel functions of the first and second kinds of order η. The quantities C_1, C_2 which appear in (4.7) and below are independent of z and are to be determined by the boundary conditions.

If $D\neq 0$ and $B=-1$, eqn (4.6) has the solution

$$\phi=\frac{(C+Dz)^{1/2}}{z}\{C_1(C+Dz)^m+C_2(C+Dz)^{-m}\}, \tag{4.9}$$

where

$$m=\frac{1}{D}[p^2+(D/2)^2]^{1/2}. \tag{4.10}$$

If $D=0$, (4.6) has a solution in terms of exponential functions, viz.

$$\phi=\frac{1}{z}\{C_1\exp(-pzC^B)+C_2\exp(pzC^B)\}. \tag{4.11}$$

If

$$\frac{2\rho(1+\mu)}{\delta^2A_2{}^6E}=G\,e^{Jx}, \tag{4.12}$$

where G, J are constants, then eqn (4.3) becomes

$$\frac{d^2\phi}{dx^2}-p^2G\,e^{Jx}\phi=0, \tag{4.13}$$

which has the solution

$$\phi=C_1K_0(p\psi)+C_2I_0(p\psi), \tag{4.14}$$

where

$$\psi=\frac{2G^{1/2}}{J}e^{Jx/2}. \tag{4.15}$$

Finally, if

$$\frac{2\rho(1+\mu)}{\delta^2A_2{}^6E}=Qz^4\,e^{Mz}, \tag{4.16}$$

where Q, M are constants, then ϕ satisfies

$$\frac{d^2\phi}{dz^2}+\frac{2}{z}\frac{d\phi}{dz}-p^2Q\,e^{Mz}\phi=0. \tag{4.17}$$

The solution to this equation is

$$\phi=\frac{1}{z}\{C_1K_0(p\lambda)+C_2I_0(p\lambda)\}, \tag{4.18}$$

where

$$\lambda=\frac{2Q^{1/2}}{M}e^{Mz/2}. \tag{4.19}$$

In each of the cases listed above both the equation that the various parameters of the shell satisfy as well as the corresponding equation for ϕ and its solution are given. Thus the results of Eason's method, as applied to the present problem, can be verified directly by showing that each expression for ϕ satisfies the associated differential equation.

The results listed above contain sufficient arbitrary constants for them to have quite a wide application. We now apply these results to some particular problems.

Consider the middle surface of the undeformed shell generated by rotating the curve

$$y=h(x),\ x\ge a>0,\ y\ge 0, \tag{4.20}$$

about the y-axis. We assume that $h(x)$ is a positive, single-valued function which is twice continuously differentiable. In the notation of (2.2), the metric coefficients for the surface are

$$A_1=[1+(h'(\alpha))^2]^{1/2},\ A_2=\alpha,\ \alpha\ge a, \tag{4.21}$$

where the curvilinear coordinate α is the radius of the circular cross-section of the surface. Thus the curves α = constant are the parallels, while the other coordinate curves are the meridians.

If the curve (4.20) is rotated about the x-axis, then the metric coefficients are

$$A_1=[1+(h'(\alpha))^2]^{1/2},\ A_2=h(\alpha),\ \alpha\ge a. \tag{4.22}$$

Here α is the coordinate measured from the origin along the axis of symmetry of the surface. Clearly, if the function $y=h$ has a unique inverse, it is not necessary to consider separately surfaces generated by this second method. For all particular examples considered in this section we shall take the metric coefficients in the form (4.21) with the single exception of the circular cylinder.

From the form of A_2 in (4.21), the condition (4.5) suggests that fairly simple solutions can be found when the thickness is proportional to α^{-3}. Thus we consider

$$\delta=\delta_0(a/\alpha)^3,\ \delta_0\ \text{constant}. \tag{4.23}$$

Choosing $K=0$ in (4.4) and B, C, D in (4.5) appropriately, we discover that (4.23) satisfies (4.5) and that ϕ is given by (4.7) with

$$\eta=-\frac{1}{2},\quad r=\frac{x}{\delta_0a^3}\left[\frac{2\rho(1+\mu)}{E}\right]^{1/2},\quad z=\frac{1}{x}. \tag{4.24}$$

To obtain an outgoing wave in the semi-infinite shell $\alpha\ge a$ we choose $C_2=0$. Hence, since the Bessel function becomes an elementary function for $\eta=-1/2$, we find that

$$\phi=C_1e^{-rp}. \tag{4.25}$$

If a torsional stress is suddenly applied at the boundary of the semi-infinite shell and thereafter steadily maintained, then

$$T(a,t) = AU(t), \tag{4.26}$$

where $U(t)$ is the Heaviside unit function defined above. Thus $\phi(x,p)$ satisfies the boundary condition

$$\phi(0,p) = AA_2^{\,2}(a)/p. \tag{4.27}$$

Hence, on applying this boundary condition to determine C_1 and then inverting the Laplace transform, we find from (4.25) that

$$T(\alpha,t) = A\left(\frac{a}{\alpha}\right)^2 U(t-L), \tag{4.28}$$

where

$$L(\alpha) = \int_a^\alpha d\tau/c(\tau) = \left[\frac{2\rho(1+\mu)}{E}\right]^{1/2} s. \tag{4.29}$$

Here $c(\alpha)$ is the speed of propagation of the stress wave and s is the arc-length along a meridian of the surface from $\alpha = a$, viz.

$$s = \int_a^\alpha [1+(h'(\tau))^2]^{1/2}\, d\tau.$$

In the case in which T takes any value $q(t)$ at $\alpha = a$ for $t > 0$, we have from Duhamel's theorem that

$$T(\alpha,t) = \left(\frac{a}{\alpha}\right)^2 U(t-L)q(t-L). \tag{4.30}$$

As well, the natural frequencies of the shell whose thickness is given by (4.23) and fixed at both ends $\alpha = a$ and $\alpha = b$ are $n\pi/L(b)$, $n = 1, 2, \ldots$.

We emphasize that the results just obtained for shells whose thickness are given by the particular form (4.23) hold for the surface obtained by rotating any curve in the form (4.20) about the y-axis.

Next consider the shell whose metric coefficient A_1 in (4.21) and thickness δ are proportional to an arbitrary power of α. That is, we consider

$$A_1 = A_0\left(\frac{\alpha}{a}\right)^\gamma, \delta = \delta_0\left(\frac{\alpha}{a}\right)^N, \tag{4.31}$$

where A_0, δ_0, γ, and N are constants. Choosing K in (4.4) and B, C, D in (4.5) appropriately, we find that (4.31) satisfies (4.5), where on determining the corresponding solution it is necessary to consider separately the cases in which $\gamma \neq -1$ and $\gamma = -1$.

If $\gamma \neq -1$, we find from (4.7) that

$$\phi = \alpha^{(\gamma+N+2)/2}\{C_1K_\eta(rp) + C_2I_\eta(rp)\}, \tag{4.32}$$

where

$$r = \frac{A_0 a}{\gamma+1}\left(\frac{\alpha}{a}\right)^{\gamma+1}, \eta = \frac{1}{2}\left(\frac{\gamma+N+4}{\gamma+1}\right). \tag{4.33}$$

Although the expression for η in (4.8) implies $\eta \neq 0$ for any finite B, the solution (4.32) holds for all η including $\eta = 0$. The validity of the solution for this particular case can be established from (4.14).

For a semi-infinite shell with a Heaviside input at the boundary, ϕ satisfies the boundary condition (4.27). Since $dr/d\alpha > 0$, we choose $C_2 = 0$ to obtain an outgoing wave, and then apply this boundary condition to determine C_1. On inverting the Laplace transform, we find from (4.32) that

$$T(\alpha, t) = \frac{A}{2\pi i}\left(\frac{\alpha}{a}\right)^{(\gamma+N)/2} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{pt} K_\eta(rp)}{p K_\eta(r_0 p)}\, dp, \tag{4.34}$$

where $\epsilon > 0$ is a constant chosen such that the integrand has no singularities in $Re(p) \geq \epsilon$, and r_0 is r evaluated at $\alpha = a$.

It is difficult to deal any further with (4.34) unless particular values of η are taken. However, one result can be obtained immediately, namely an expansion of T valid near the wavefront. Such an expansion can be obtained (according to standard methods) by substituting into (4.34) the expansion of the integrand for large p. Working out the first three terms in the wavefront expansion, we obtain

$$\begin{aligned} T(\alpha, t) = A\left(\frac{\alpha}{a}\right)^{(N-1)/2} U[t - L(\alpha)]\Bigg\{1 &+ \left(\frac{1}{r} - \frac{1}{r_0}\right)\left(\frac{4\eta^2 - 1}{8}\right)(t - L) \\ &+ \left(\frac{1}{r} - \frac{1}{r_0}\right)\left(\frac{4\eta^2 - 1}{256}\right)\left(\frac{4\eta^2 - 9}{r} - \frac{4\eta^2 + 7}{r_0}\right)(t - L)^2 + 0[(t - L)^3]\Bigg\}. \end{aligned} \tag{4.35}$$

We may also obtain some particularly simple results from (4.34) when η is half of an odd integer. Under these circumstances $K_\eta(x)$ becomes an elementary function and the Laplace transform can be inverted explicitly. For example, if $\eta = \pm 1/2$, then

$$T(\alpha, t) = A\left(\frac{\alpha}{a}\right)^{(N-1)/2} U(t - L). \tag{4.36}$$

As well, if $\eta = \pm 3/2$, then

$$T(\alpha, t) = A\left(\frac{\alpha}{a}\right)^{(N-2\gamma-3)/2} U(t - L)\left[1 + \left(\frac{r}{r_0} - 1\right)\exp\left\{-\frac{1}{r_0}(t - L)\right\}\right], \tag{4.37}$$

while if $\eta = \pm 5/2$, then

$$\begin{aligned} T(\alpha, t) = A\left(\frac{\alpha}{a}\right)^{(N-4\gamma-5)/2} U(t - L)\Bigg\{1 &+ \left(\frac{r}{r_0} - 1\right)\Bigg[\left(1 + \frac{r}{r_0}\right)\cos\left(\frac{\sqrt{(3)}}{2}\frac{t - L}{r_0}\right) \\ &- \left(\frac{r}{r_0} - 1\right)\sqrt{(3)}\sin\left(\frac{\sqrt{(3)}}{2}\frac{t - L}{r_0}\right)\Bigg]\exp\left(-\frac{3}{2}\frac{t - L}{r_0}\right)\Bigg\}. \end{aligned} \tag{4.38}$$

In the event that the shell is bounded by the two curves $\alpha = a$ and $\alpha = b$, and clamped at $\alpha = b$, we find for an impact described by (4.26) that

$$T(\alpha, t) = \frac{A}{2\pi i}\left(\frac{\alpha}{a}\right)^{(\gamma+N)/2} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{pt}\Psi_p(r, r_1)}{p\Psi_p(r_1, r_0)}\, dp. \tag{4.39}$$

Here r_0 is r evaluated at $\alpha = a$, r_1 is r evaluated at $\alpha = b$, and

$$\Psi_p(r, r_1) = K_\eta(r_1 p) I_\eta(rp) - I_\eta(r_1 p) K_\eta(rp). \tag{4.40}$$

The function $\Psi_p(r_1, r_0)$ is an entire function of p with simple zeros at $p = \pm i\omega_n$ $(n = 1, 2, \ldots)$, where ω_n are the positive roots of the equation

$$Y_\eta(r_1\omega)J_\eta(r_0\omega) - Y_\eta(r_0\omega)J_\eta(r_1\omega) = 0, \tag{4.41}$$

viz. the natural frequencies of the shell. In this equation $J_\eta(x)$ and $Y_\eta(x)$ are the ordinary Bessel functions of the first and second kinds.

From (4.39), we may obtain a representation of the stress in terms of the torsional modes of the shell by means of the calculus of residues. If we close the contour to the left and sum the residues, we obtain

$$T(\alpha,t)=A\left(\frac{\alpha}{a}\right)^{(\gamma+N)/2}\left\{\frac{r^{2\eta}-r_1^{2\eta}}{r_0^{2\eta}-r_1^{2\eta}}\left(\frac{r_0}{r}\right)^{\eta}\right.$$
$$\left.+\pi\sum_{n=1}^{\infty}\left[\frac{J_\eta(r_1\omega_n)Y_\eta(r\omega_n)-Y_\eta(r_1\omega_n)J_\eta(r\omega_n)}{J_\eta^2(r_1\omega_n)-J_\eta^2(r_0\omega_n)}\right]\times J_\eta(r_0\omega_n)J_\eta(r_1\omega_n)\cos\omega_n t\right\}. \tag{4.42}$$

This expression is not suitable for discoverning the location of the wavefronts and the magnitude of the discontinuities across these fronts. The contour integral representation (4.39) is more appropriate for this purpose as shall be demonstrated in the appendix. However, if η is half an odd integer the wave character of the solution (4.42) becomes transparent. For example, if $\eta=1/2$, then

$$T(\alpha,t)=A\left(\frac{\alpha}{a}\right)^{(N-1)/2}\left[1-\frac{L(\alpha)}{L(b)}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\sin\frac{n\pi}{L(b)}[t+L(\alpha)]\right.\right.$$
$$\left.\left.-\sin\frac{n\pi}{L(b)}[t-L(\alpha)]\right\}\right]. \tag{4.43a}$$

The Fourier sine series in (4.43a) can be summed and we find that

$$T(\alpha,t)=A\left(\frac{\alpha}{a}\right)^{(N-1)/2}\left\{1-\frac{L(\alpha)}{L(b)}+F[t-L(\alpha)]-F[t+L(\alpha)]\right\}, \tag{4.43b}$$

where

$$F(x)=\frac{1}{2}\left[1-\frac{x}{L(b)}\right], 0\le x\le L(b), \tag{4.44}$$

and F is defined outside $[0, L(b)]$ by its odd periodic extension.

The solution (4.34) for the semi-infinite shell was found provided $\gamma\neq-1$. The corresponding solution when $\gamma=-1$ can be found from (4.9). After choosing the constants appropriately, applying the boundary conditions, and then inverting the transform, we find that

$$T(\alpha,t)=A\left(\frac{\alpha}{a}\right)^{(N-1)/2}U(t-L)\left\{1-\left(\frac{N+3}{2}\right)\ln\left(\frac{\alpha}{a}\right)\int_L^t\frac{J_1(h\sqrt{\tau^2-L^2})\,d\tau}{\sqrt{\tau^2-L^2}}\right\}. \tag{4.45}$$

In this formula,

$$h=\frac{N+3}{2A_0}\left[\frac{E}{2\rho(1+\mu)}\right]^{1/2}, \tag{4.46}$$

and the phase function $L(\alpha)$ is given by

$$L(\alpha)=\left[\frac{2\rho(1+\mu)}{E}\right]^{1/2}A_0\ln\left(\frac{\alpha}{a}\right). \tag{4.47}$$

The expression for T given by (4.45) can be expanded for small values of $(t-L)$, i.e. near the wavefront. Calculating the first four terms of such an expansion we have

$$T(\alpha,t)=A\left(\frac{\alpha}{a}\right)^{(N-1)/2}U(t-L)\left\{1-\frac{Lh^2}{2}(t-L)+\frac{L^2h^4}{16}(t-L)^2\right.$$
$$\left.+\frac{Lh^4}{48}\left(1-\frac{L^2h^2}{48}\right)(t-L)^3+O[(t-L)^4]\right\}. \tag{4.48}$$

In the specific model just considered the metric coefficient A_1 was assumed to have the two parameter form given in (4.31). Although this form is particularly simple, in general the corresponding surface cannot be easily constructed. From (4.20), we see that the middle surface of the undeformed shell is obtained by rotating the curve

$$y = \int \left[A_0^2 \left(\frac{x}{a}\right)^{2\gamma} - 1 \right]^{1/2} dx, x \geq a,$$

about the y-axis, where A_0 must be suitably restricted to give real values of y. In general this curve is not described by an elementary function although for particular choices of the constants it is. For example, if $\gamma = 0$ and $A_0 = \csc \theta$ we obtain a cone with semivertex angle θ. If $\gamma = 1/2$ we obtain the surface of revolution whose profile is

$$y = \frac{2a}{3A_0} \left(A_0^2 \frac{x}{a} - 1 \right)^{3/2}, x \geq a, A_0 \geq 1.$$

In view of this last remark, we shall now confine our attention to specific simple shells of revolution, namely conical and cylindrical shells. The results of Eason's method will be used to discover solutions to impact problems for such shells when the thickness is expressed in terms of exponential or logarithmic functions. For a conical shell of semivertex angle θ, $h(x) = x \cot \theta$ in (4.21), while for a cylindrical shell of radius R, $h(x) = R$ in (4.22). Hence the metric coefficients for a conical shell are

$$A_1 = \csc \theta, A_2 = \alpha, \tag{4.49a}$$

and for a cylindrical shell are

$$A_1 = 1, A_2 = R. \tag{4.49b}$$

If we consider a semi-infinite conical shell of thickness

$$\delta = \delta_0 \left(\frac{a}{\alpha}\right)^3 \exp\left\{ N\left(\frac{\alpha}{a} - 1\right) \right\}, \tag{4.50a}$$

or a semi-infinite cylindrical shell of thickness

$$\delta = \delta_0 \exp\left\{ N\left(\frac{\alpha}{a} - 1\right) \right\}, \tag{4.50b}$$

subject to an impact described by the boundary condition (4.26), then we find from (4.9) that

$$T(\alpha, t) = A\left[\frac{A_2(a)\delta(\alpha)}{A_2(\alpha)\delta(a)}\right]^{1/2} U[t - L(\alpha)] \left\{ 1 - hL \int_L^t \frac{J_1(h\sqrt{(\tau^2 - L^2)})\, d\tau}{\sqrt{(\tau^2 - L^2)}} \right\}, \tag{4.51}$$

where

$$h = \frac{1}{2} \left[\frac{E}{2\rho(1+\mu)} \right]^{1/2} \frac{N}{aA_1}, L(\alpha) = A_1 \left[\frac{2\rho(1+\mu)}{E} \right]^{1/2} (\alpha - a). \tag{4.52}$$

This solution can be expanded for small values of $(t - L)$ to obtain a wavefront expansion of the same form as (4.48). The natural frequencies for the conical and cylindrical shells whose thickness are given by (4.50a) and (4.50b) respectively are $\{h + [n\pi/L(b)]^2\}^{1/2}$, $(n = 0, 1, 2, \ldots)$.

Similarly, we may use other results of Eason's method when δ is expressed in some other form. If the conical shell has thickness

$$\delta = \delta_0 \left(\frac{a}{\alpha}\right)^4 \left(\ln \frac{\alpha}{a} e \right)^{-2}, \tag{4.53a}$$

or the cylindrical shell has thickness

$$\delta = \delta_0\left(\frac{a}{\alpha}\right)\left(\ln\frac{\alpha}{a}e\right)^{-2}, \tag{4.53b}$$

then we find from (4.18) that the solution to the impact problem for such semi-infinite shells is

$$T(\alpha, t) = \frac{A}{2\pi i}\left[\frac{\alpha A_2(a)\delta(\alpha)}{aA_2(\alpha)\delta(a)}\right]^{1/2}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{e^{pt}K_0(rp)\,dp}{pK_0(r_0p)}. \tag{4.54}$$

Here

$$r = \sqrt{\left(\frac{2\rho(1+\mu)}{E}\right)}A_1\alpha, \tag{4.55}$$

and r_0 is r evaluated at $\alpha = a$. The wavefront expansion for this solution can be found in the same way as was done for (4.34). The frequency equation for these shells is (4.41) with $\eta = 0$.

REFERENCES

1. H. Garnet, M. A. Goldberg and V. L. Salerno, Torsional vibrations of shells of revolution. *J. Appl. Mech.* **28**, 571 (1961).
2. E. Mathieu, Mémoire sur le mouvement vibratoire des cloches. *J. Ec. Polytech.* **51**, 177 (1882).
3. A. E. H. Love, The small free vibrations and deformation of a thin elastic shell. *Philos. Trans. R. Soc. Lond.* **179**, 491 (1888).
4. H. Lamb, On the vibrations of a spherical shell. *Proc. Lond. Math. Soc.* **14**, 50 (1882).
5. S. R. Soni, R. K. Jain and C. Prasad, Torsional vibrations of shells of revolution of variable thickness. *J. Acoust. Soc. Amer.* **53**, 1445 (1973).
6. V. Ramamurti and N. Ganesan, Torsional vibration of prismatic shells. *J. Sound Vibration* **38**, 195 (1975).
7. R. M. Lewis and J. B. Keller, Asymptotic methods for partial differential equations; the reduced wave equation and Maxwell's equations. Courant Institute of Mathematical Sciences, Report EM-194, N.Y.U. (1964).
8. D. W. Barclay and T. B. Moodie, Verifications of a formal technique for viscoelastodynamics. *Bull. Austral. Math. Soc.* **11**, 279 (1974).
9. G. Eason, Wave propagation in inhomogeneous elastic media, solution in terms of Bessel functions. *Acta Mech.* **7**, 137 (1969).
10. V. V. Novozhilov, *The Theory of Thin Shells*. Noordhoff, Leydin (1959).
11. A. Erdélyi, *Asymptotic Expansions*. Dover, New York (1956).

APPENDIX A

In this appendix we shall verify the validity of the leading term in the expansion obtained by the formal asymptotic procedure for an impact applied to the boundary $\alpha = a$ of a finite shell of revolution, initially quiescent and bounded by the curves $\alpha = a$ and $\alpha = b$. We shall suppose that the stress T satisfies the impact boundary condition (4.26) at $\alpha = a$, while the end $\alpha = b$ is maintained in a stress-free state. Thus $\phi(x, p)$ satisfies the boundary conditions

$$\phi(o, p) = AA_2^2(a)/p, \quad \phi(l, p) = 0, \tag{A1}$$

where

$$l = \int_a^b \delta(\tau)A_1(\tau)A_2^3(\tau)\,d\tau. \tag{A2}$$

By the Laplace inversion theorem, the solution to (2.4) can be represented by

$$T(\alpha, t) = \frac{1}{2\pi i A_2^2(\alpha)}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi(x, p)\,dp. \tag{A3}$$

As is well known, the wavefront expansion for T can be extracted from (A3) by appealing to the behaviour of the integrand for large values of the Laplace transform parameter p. In connection with this idea we state the following lemma, which can be easily proved.

Lemma. Suppose that $\phi(x, p)$ is analytic in Re $(p) \geq \epsilon > 0$ and that

$$\phi(x, p) = \frac{A(x)}{p}\exp\{-pL(x)\}\left[\sum_{n=0}^{N}\frac{A_n(x)}{p^n} + O(p^{-N-1})\right],$$

as $p \to \infty$ in $\text{Re}(p) \geq \epsilon > 0$, then

$$\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi(x, p)dp = A(x)U(t-L)\left\{\sum_{n=0}^{N}\frac{A_n(x)(t-L)^n}{n!} + O[(t-L)^{N+1}]\right\}$$

as $(t - L) \to 0$.

In view of this lemma, the first step in determining the wave-front expansion is to discover the asymptotic form of solutions to the ordinary differential eqn (4.3). To discover this asymptotic behaviour, we apply the methods described by Erdélyi[11]. Make the transformation

$$Z(x) = p\left[\frac{2\rho(1+\mu)}{E}\right]^{1/2}\int_0^x \frac{dW}{\delta(\xi)A_2^{\,3}(\xi)},\ \phi(x,p) = [Z'(x)]^{1/2}V \tag{A4}$$

which carries (4.3) into

$$\frac{d^2V}{dZ^2} - V = \frac{\Lambda(x,p)}{Z'(x)}V. \tag{A5}$$

The variables W and ξ in (A4) are related by

$$W = \int_a^{\xi} \delta(\tau)A_1(\tau)A_2^{\,3}(\tau)\,d\tau, \tag{A6}$$

while the function Λ which appears in (A5) is given by

$$\Lambda(x,p) = \frac{1}{Z'}\left\{\frac{1}{2}\frac{d}{dx}\left(\frac{Z''}{Z'}\right) - \frac{1}{4}\left(\frac{Z''}{Z'}\right)^2\right\}, \tag{A7}$$

where prime denotes differentiation with respect to x. Clearly, $\Lambda(x,p)$ is an analytic function of p in $\mathrm{Re}(p) \geq \epsilon > 0$ and

$$\Lambda(x,p) = O(1/p), \tag{A8}$$

uniformly in $0 \leq x \leq l$.

If we apply the method of variation of parameters to (A5), temporarily regarding the right hand side as a known function, we obtain the integral equation

$$V(x) = C_1 e^{Z(x)} + C_2 e^{-Z(x)} + \int_{x_0}^{x} \sinh[Z(x) - Z(\tau)]\Lambda(\tau,p)V(\tau)\,d\tau, \tag{A9}$$

where x_0 is some fixed point in $0 \leq x \leq l$, C_1 and C_2 are independent of x, and V is regarded as a function of x. Equation (A9) is a Volterra integral equation; the existence and differentiability of the solution can be demonstrated by constructing the solution by the method of successive approximations. The uniform convergence of this iterative procedure will ensure that the solution is an analytic function of p in $\mathrm{Re}(p) > 0$, provided C_1 and C_2 are analytic in $\mathrm{Re}(p) > 0$. By differentiating the integral equation, it can be verified that ϕ given in (A4) satisfies the differential eqn (4.3), where V in (A4) is the solution to the integral equation. As well, in view of the bound (A8), we may apply standard methods to determine the asymptotic behaviour of the solution to the integral equation for any choice of C_1 and C_2. Thus in turn we obtain the behaviour of solutions to the differential equation.

If we choose $C_1 = 0$, $C_2 = p^{1/2}$, and $x_0 = l$, we find that (4.3) has a solution $\phi_1(x,p)$ which is analytic in $\mathrm{Re}(p) \geq \epsilon > 0$ and such that

$$\phi_1(x,p) = [c(\alpha)A_1(\alpha)\delta(\alpha)A_2^{\,3}(\alpha)]^{1/2}e^{-pL(\alpha)}[1 + O(1/p)], \tag{A10}$$

uniformly in $0 \leq x \leq l$. In this formula, $c(\alpha)$ is the variable stress wave velocity,

$$c(\alpha) = \frac{1}{A_1(\alpha)}\left[\frac{E}{2\rho(1+\mu)}\right]^{1/2}, \tag{A11}$$

while $L(\alpha)$ is the phase function

$$L(\alpha) = \int_a^{\alpha}\frac{d\tau}{c(\tau)}. \tag{A12}$$

If we choose $C_1 = p^{1/2}$, $C_2 = 0$, and $x_0 = 0$, then we obtain a second linearly independent solution $\phi_2(x,p)$ whose asymptotic behaviour is found from (A10) with L replaced by $-L$ in the exponential function.

If we express $\phi(x,p)$ in terms of the linearly independent pair ϕ_1, ϕ_2 and apply the boundary conditions (A1), we find that

$$\phi(x,p) = AA_2^{\,2}(a)[\phi_2(l,p)\phi_1(x,p) - \phi_1(l,p)\phi_2(x,p)]/p\psi(p), \tag{A13}$$

where

$$\psi(p) = \phi_1(o,p)\phi_2(l,p) - \phi_1(l,p)\phi_2(o,p). \tag{A14}$$

The solution to the initial-boundary value problem is then given by (A3) with the expression (A13) for ϕ. This solution has a sequence of discontinuities at the successive outgoing wavefronts from $\alpha = a$ and at the successive fronts reflected from the boundary at $\alpha = b$. The location and magnitude of these jumps are discovered by examining (A13) for large values of p.

By appealing to the asymptotic behaviour of ϕ_1 and ϕ_2, we find that

$$AA_2^{\,2}(a)\phi_2(l,p)\phi_1(x,p)/p\psi(p) = \frac{AA_2^{\,2}(a)}{2p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^{\,3}(\alpha)}{c(a)A_1(a)\delta(a)A_2^{\,3}(a)}\right]^{1/2}\frac{\exp\{p[L(b) - L(\alpha)]\}}{\sinh\{pL(b)\}}[1 + O(1/p)], \tag{A15}$$

$$AA_2^2(a)\phi_1(l,p)\phi_2(x,p)/p\psi(p) = \frac{AA_2^2(a)}{2p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^3(\alpha)}{c(a)A_1(a)\delta(a)A_2^3(a)}\right]^{1/2}\frac{\exp\{p[L(\alpha)-L(b)]\}}{\sinh\{pL(b)\}}[1+O(1/p)]. \tag{A16}$$

In view of these formulas and the bound

$$|\sinh\{pL(b)\}| \geq A_0 \exp\{\epsilon L(b)\}, \operatorname{Re}(p) \geq \epsilon > 0,$$

where A_0 is a constant, we may choose $\epsilon > 0$ such that $\phi(x, p)$ has no singularities in $\operatorname{Re}(p) \geq \epsilon$. As well, on employing the identity

$$(\sinh z)^{-1} = 2e^{-z} + e^{-2z}(\sinh z)^{-1}, \tag{A17}$$

to the hyperbolic sine appearing in (A15) and (A16), we may write (A13) as

$$\phi(x,p) = \sum_{k=0}^{4} \phi^k(x,p). \tag{A18}$$

The functions $\phi^k(x,p)$ are analytic and from (A15)

$$\phi^1(x,p) = \frac{AA_2^2(a)}{p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^3(\alpha)}{c(a)A_1(a)\delta(a)A_2^3(a)}\right]^{1/2}\exp\{-pL(\alpha)\}[1+O(1/p)], \tag{A19}$$

$$\phi^3(x,p) = \frac{AA_2^2(a)}{2p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^3(\alpha)}{c(a)A_1(a)\delta(a)A_2^3(a)}\right]^{1/2}\frac{\exp\{-p[L(b)+L(\alpha)]\}}{\sinh\{pL(b)\}}[1+O(1/p)], \tag{A20}$$

while from (A16)

$$\phi^2(x,p) = \frac{-AA_2^2(a)}{p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^3(\alpha)}{c(a)A_1(a)\delta(a)A_2^3(a)}\right]^{1/2}\exp\{p[L(\alpha)-L(b)]\}[1+O(1/p)], \tag{A21}$$

$$\phi^4(x,p) = \frac{-AA_2^2(a)}{2p}\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2^3(\alpha)}{c(a)A_1(a)\delta(a)A_2^3(a)}\right]^{1/2}\frac{\exp\{p[L(\alpha)-3L(b)]\}}{\sinh\{pL(b)\}}[1+O(1/p)]. \tag{A22}$$

Thus, from (A3) we may write the stress field as

$$T(\alpha,t) = T^1(\alpha,t) + T^2(\alpha,t) + \frac{1}{2\pi i A_2^2(\alpha)}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi^3(x,p)\,dp + \frac{1}{2\pi i A_2^2(\alpha)}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi^4(x,p)\,dp, \tag{A23}$$

where

$$T^1(\alpha,t) = \frac{1}{2\pi i A_2^2(\alpha)}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi^1(x,p)\,dp, \tag{A24}$$

$$T^2(\alpha,t) = \frac{1}{2\pi i A_2^2(\alpha)}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{pt}\phi^2(x,p)\,dp. \tag{A25}$$

In view of the asymptotic formulae for ϕ^1 and ϕ^2, and the lemma stated at the beginning of the Appendix, we have

$$T^1(\alpha,t) = A\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2(a)}{c(a)A_1(a)\delta(a)A_2(\alpha)}\right]^{1/2} U[t-L(\alpha)][1+0(t-L)], \tag{A26}$$

$$T^2(\alpha,t) = A\left[\frac{c(\alpha)A_1(\alpha)\delta(\alpha)A_2(a)}{c(a)A_1(a)\delta(a)A_2(\alpha)}\right]^{1/2} U[t+L(\alpha)-L(b)]\times\{1+0[t+L(\alpha)-L(b)]\}. \tag{A27}$$

These formulae are the same as those obtained by the formal method for the first incident and reflected wave. In addition, if we examine the remaining two integrals on the right of (A23), we find that the first vanishes for $t - L(\alpha) < 2L(b)$ and has jumps at $t - L(\alpha) = 2nL(b)$, $(n = 1, 2, \ldots)$, the remaining wavefronts incident on the boundary at $\alpha = b$. The magnitude of the discontinuities across these fronts can be found by repeatedly applying the identity (A17) to the hyperbolic sine which appears in (A20). Similarly, the last integral on the right of (A23) vanishes for $t + L(\alpha) < 4L(b)$, and has jumps at $t + L(\alpha) = 2nL(b)$, $(n = 2, 3, \ldots)$, the successive wavefronts reflected from the boundary at $\alpha = b$.